J. Fluid Mech. (1969), *vol.* 36, *part* 3, *pp.* 585–603
Printed in Great Britain

Sound generation by turbulent two-phase flow

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(Received 21 November 1968)

Sound generation by turbulent two-phase flow is considered by the methods of Lighthill's theory of aerodynamic noise. An inhomogeneous wave equation is derived, in which the effects of one phase on the other are represented by monopole, dipole and quadrupole distributions. The resulting power outputs are obtained for the case of a distribution of small air bubbles in water. The monopole radiation resulting from volumetric response of the bubbles to the turbulent pressure field overwhelms that from the quadrupoles equivalent to the turbulent flow, the increase in acoustic power output being about 70 dB for a volume concentration of 10 %. The monopole radiation occurs through the forced response of the bubbles at the turbulence frequency; resonant response is shown to be impossible when the excitation is due to turbulence alone. Surface radiation arises from the edge of a cloud of bubbles. This radiation is important when the region containing bubbles is in the form of a sheet with thickness smaller than the length scale of the turbulent motion. Dipole radiation is also considered, and found to be negligible whenever monopole sources are present. In the case of a dusty gas, only dipole and quadrupole sources are present, and here it is shown that the dipole radiation is equivalent to an increase in the usual quadrupole radiation. The increase depends upon the mass concentration of dust, and is significant for mass concentrations in excess of unity.

1. Introduction

In this paper we consider the sound radiation from a finite region of turbulent or unsteady flow, in which the fluid consists of a mixture of two phases. For the most part attention is confined to the case of a small volume concentration of air bubbles in water, though the case of a gas containing small dust particles is also examined briefly. Much work has been done in the past on the radiation from a single air bubble in water (e.g. Strasberg 1956) when various forms of excitation are responsible for the motion of the bubble. The bulk properties of a distribution of bubbles in water have also been studied, in particular the well-known drastic reduction of the sound speed caused by even a very small concentration of bubbles, and the variation of the sound speed with frequency. A review of these, and many other effects, is given by Batchelor (1967). Much less has been done on the excitation of a single bubble, or a distribution of bubbles, by a turbulent pressure field. This problem is discussed here on the lines of the Lighthill (1952) theory of aerodynamic sound generation.

A Lighthill inhomogeneous wave equation is first derived, in which the action of the bubbles on the water is represented by an equivalent distribution of monopole and dipole sources, in addition to the quadrupoles acoustically equivalent to the fluctuating flow. When no boundaries are present in the flow, the acoustic power output can be found in terms of the source strengths by the usual formulae. In order to estimate the monopole source strength, the response of a single bubble in turbulent flow is then considered using familiar equations. The pressure spectrum of a turbulent flow is relatively broad, and there is the possibility that large changes in bubble volume may arise from the small spectral intensity of the pressure at the high natural resonance frequency of the bubble. This would be a difficult effect to estimate reliably, for, although it is possible to give an analytical form for the pressure spectrum at high frequencies using the Kolmogorov theory of the fine-scale structure of turbulence (Batchelor 1953), the resonance would be limited only by dissipative effects whose character is not yet properly understood. In particular, the radiation damping of a bubble at resonance when surrounded by a distribution of bubbles is difficult to analyze, since the sound speed at high frequency in the distribution is complex, and varies with frequency. A detailed consideration of these effects is, fortunately, not necessary here, for the possibility of significant resonant response under excitation by turbulence alone is ruled out (§ 5). The reason for this is that the length scale over which the pressure field remains coherent at the resonance frequency is found to be very small compared with the bubble radius. The phase of the pressure field then varies rapidly over the bubble surface, whereas significant volume response requires the pressure to be substantially in phase all over the surface.

For this reason, the extension given by Curle (1955) to the Lighthill theory, taking account of the effect of surfaces in the flow, is not considered. The only way in which the presence of surfaces can alter the inferences to be made about the effect of bubbles on the radiated noise, is by introducing the possibility of coherent forcing, at the resonance frequency, over length scales large compared with the bubble radius. If the behaviour of the surface is controlled entirely by the turbulent flow, this possibility is again ruled out, since the length scales of the forcing due to the surface would be of the order of those in the turbulent flow itself. If, however, the motion of the surface were controlled by some *external* means, we could have the possibility of coherent forcing at the resonance frequency. This is exactly what happens if, for example, the bubble is irradiated by a sound wave generated by motion of a surface (Hunter 1967). Even then, this does not necessarily mean that resonant response is significant, in view of the high dissipation occurring in a distribution of bubbles at the resonant frequency. If such cases, in which the control of the surface behaviour by external means provides a length scale large compared with the bubble radius, are excluded, we can entirely discount the resonant response of the bubbles, and no further attention need be paid to the effect of surfaces.

Certain effects of two-phase flow are obvious, and will receive no more attention in this paper. These concern surface and volume sources in an infinite region of bubbly fluid in which the sound speed c_m is significantly lower than the sound speed c_α in pure water. According to the usual ideas of aerodynamic noise theory, the intensities of monopole, dipole and quadrupole sources vary as c^{-1}, c^{-3} and

c^{-5}, where c is the sound speed in the far-field of the sources. Therefore, in this case, the power output of these sources will be increased by the factors c_α/c_m, $(c_\alpha/c_m)^3$ and $(c_\alpha/c_m)^5$ respectively, over their values for emission into pure water. However, in practice this case never arises, and one is usually concerned with situations in which the bubbly liquid occupies a region with typical dimension small compared with a sound wavelength in pure water. The theory is therefore set up in a form capable of handling these cases where the fluid mixture is inhomogeneous on scales smaller than a wavelength. In this way, changes in the turbulence-generated sound are attributed to a distribution of acoustic sources, whereas the increases noted above for the infinite bubbly region are essentially connected with sound propagation over distances of many wavelengths. The physical bases for the results in the two cases are thus quite different.

In the formulation given here, monopole sources of sound arise from the forced response of the bubbles at the frequency characteristic of the turbulence. They lead to an efficiency proportional to the fifth power of Mach number, which is the variation usually ascribed to quadrupole sources. In fact it is shown that the monopole intensity is just that of the usual Lighthill quadrupoles, but augmented by the factor $(c_\alpha/c_m)^4$, which should be contrasted with the $(c_\alpha/c_m)^5$ factor referred to previously. c_α/c_m can easily exceed 10, so that the presence of bubbles in a turbulent flow will very greatly increase the acoustic power output. For the extreme case of a 10 % concentration of bubbles by volume the acoustic power may be increased by about 70 dB.

Apart from effects arising from the variation of bubble volume, there is the question of whether abrupt changes in the mean concentration can produce appreciable sound. The sources corresponding to a discontinuous rise in concentration are examined in § 6, where it is shown that the radiated field can be expressed in terms of a surface distribution over the interface across which the concentration changes. The radiation produced is shown to be equal to that produced by distributed sources in a volume which has one typical dimension equal to the turbulence length scale.

Dipole sources of sound arising from bubble response are also considered. As expected, they are much less efficient than the monopoles at the very low Mach numbers typical in underwater applications. The case of a dusty gas is then dealt with, in which monopole radiation cannot occur, and in which the action of the dust particles on the gas is represented entirely by a dipole distribution. Again it is shown that the presence of dust is to augment the usual quadrupole radiation. The increase in power output is less startling than that caused by bubbles, but is appreciable when the *mass* concentration of dust exceeds unity.

2. Lighthill equation for flow of air bubbles in water

We consider a finite region in which unsteady or turbulent flow occurs, and in which the fluid is a mixture of water (α-phase), and a small concentration by volume of gas bubbles (β-phase). The small quantity $\beta(\mathbf{x}, t)$ is the fraction of unit volume of the mixture which is occupied by the bubbles. ρ^α, ρ^β are the actual densities of the two phases, i.e. ρ^α = (mass of α-phase)/(volume occupied by

α-phase). The mass of α-phase in unit volume of mixture is then $(1-\beta)\rho^{\alpha}$, and the total mass per unit volume is $(1-\beta)\rho^{\alpha}+\beta\rho^{\beta}$. Far from the turbulent region $\beta = 0$, and the fluid is entirely α-phase, at rest apart from small velocities induced by the passage of sound waves from the turbulence.

We choose to formulate a Lighthill equation for the density ρ^{α}. This has the advantage of displaying clearly the action of one phase on the other in terms of acoustic sources with a simple physical interpretation. In particular, monopole and dipole distributions appear, representing the effects of mass and momentum injection into the α-phase resulting from the motion in the β-phase. The same kinds of sources appear if we consider the density $(1-\beta)\rho^{\alpha}$ instead of just ρ^{α}, but their interpretation is not quite so simple, and they are less easy to calculate. The alternative is to regard the fluid as a mixture, with density $\rho = (1-\beta)\rho^{\alpha}+\beta\rho^{\beta}$. In this case, a conventional Lighthill equation can be derived, involving quadrupole sources only. The physical interpretation is then largely lost, and the task of relating the quadrupole strength to the flow and phase parameters is difficult, as so much is hidden, for example, in the term $p-c_{\alpha}^{2}\rho$.

We are assuming the concentration β and the bubble radius a to be so small that meaningful values can be attached to the velocity and stress in the α-phase at all points $(\mathbf{x},t)$. Let u_i^{α} denote the velocity in the α-phase. Mass conservation for this phase is expressed by

$$\frac{\partial}{\partial t}(1-\beta)\rho^{\alpha}+\frac{\partial}{\partial x_j}(1-\beta)\rho^{\alpha}u_j^{\alpha} = 0,$$

which we write in the form

$$\frac{\partial}{\partial t}\rho^{\alpha}+\frac{\partial}{\partial x_j}\rho^{\alpha}u_j^{\alpha} = Q. \tag{2.1}$$

Here

$$Q = -\rho^{\alpha}\left(\frac{\partial}{\partial t}+u_j^{\alpha}\frac{\partial}{\partial x_j}\right)\ln(1-\beta)$$

$$= -\rho^{\alpha}\frac{D_{\alpha}}{Dt}\ln(1-\beta)$$

is the effective rate of mass injection density into phase α. If F_i denotes the interphase force density, the momentum equation for phase α reads

$$\frac{\partial}{\partial t}(1-\beta)\rho^{\alpha}u_i^{\alpha}+\frac{\partial}{\partial x_j}\{(1-\beta)\rho^{\alpha}u_i^{\alpha}u_j^{\alpha}+p_{ij}\} = F_i.$$

p_{ij} is the stress tensor, and is composed partly of stresses set up by the eddy motion in the α-phase, and partly of stresses set up by the response of the β-phase to the fluctuating eddy pressures. For the present there is no need to attempt to specify F_i further. We rewrite the momentum equation in the form

$$\frac{\partial}{\partial t}\rho^{\alpha}u_i^{\alpha}+\frac{\partial}{\partial x_j}\{(1-\beta)\rho^{\alpha}u_i^{\alpha}u_j^{\alpha}+p_{ij}\} = G_i, \tag{2.2}$$

where

$$G_i = F_i+G_i', \quad G_i' = (\partial/\partial t)\beta\rho^{\alpha}u_i^{\alpha}.$$

By cross-differentiation of (2.1) and (2.2) we get the required Lighthill equation, provided we note that far from the turbulent region this equation must reduce to the homogeneous wave equation

$$\{(\partial^2/\partial t^2) - c_\alpha^2 \nabla^2\}\rho^\alpha = 0,$$

where c_α is the sound speed in pure α-phase. This gives

$$\left(\frac{\partial^2}{\partial t^2} - c_\alpha^2 \nabla^2\right)\rho^\alpha = \frac{\partial Q}{\partial t} - \frac{\partial G_i}{\partial x_i} + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{2.3}$$

in which $$T_{ij} = (1-\beta)\rho^\alpha u_i^\alpha u_j^\alpha + p_{ij} - c_\alpha^2 \rho^\alpha \delta_{ij}.$$

The process of sound generation by the turbulent flow is accomplished by three distinct mechanisms. First, by a distribution of monopoles, of strength Q, equal to the rate of mass injection into the α-phase. Secondly, by a distribution of dipoles, of strength G_i. G_i is the effective force on the α-fluid, composed in part of the interphase force F_i, and in part of the term G_i'. The latter represents the momentum defect arising from the fact that a fraction β of the total volume is not occupied by α-phase. Finally, we have a distribution of quadrupoles of the Lighthill type, of strength T_{ij}. As usual, T_{ij} is dominated by the Reynolds stress terms, since, by the definition of c_α, the fluctuations in p and $c_\alpha^2 \rho^\alpha$ cancel, approximately. Viscous contributions to p_{ij} are neglected here, just as usual. In general it is quite adequate, for the order-of-magnitude arguments to be used later, to approximate T_{ij} by $\rho_0^\alpha u_i^\alpha u_j^\alpha$, where the zero suffix implies an average value.

The Mach number in typical underwater applications of flow noise theory is extremely small when based on c_α (10^{-2} at most), and the usual arguments would therefore indicate that monopole sources overwhelm the dipoles, while these in turn are very much more efficient than the quadrupoles. However, in the present problem we have a great range of new parameters: for example, the radius and resonance frequency of the bubbles, the strength of the interphase force, the relaxation time for response of the bubbles to the α-motion, and the concentration β. The usual rank ordering of acoustic sources may therefore only be valid for certain restricted ranges of the above parameters. It is the object of subsequent sections to determine how the efficiency of each type of source varies with these parameters, as well as with the parameters (length and velocity scales) of the turbulent motion.

3. Volumetric response of a bubble to a fluctuating pressure field

In this section, we consider the volumetric response of a single bubble, immersed in infinite compressible fluid, when a fluctuating pressure field is set up in the fluid. The pressure will be regarded as uniform in space far from the bubble, though fluctuating in time. A real pressure field, with finite length scale, will behave in this way provided the bubble diameter is small compared with the length scale of pressure variation. Viscous forces and thermal diffusion effects will be neglected, with the consequence that radiation damping is the only form of dissipation which limits the response of the bubble at its resonance frequency. It will be seen in § 5 that resonant response cannot occur, and therefore that the validity of this assumption is only an academic matter for our purposes.

The object is to determine the variation of bubble volume, and of the pressure scattered by the bubble, with the imposed pressure variation. The equations governing the response are well known (see, e.g. Strasberg 1956), so that only a brief derivation need be given here. In the undisturbed state, the bubble has internal pressure $\tilde{\omega}$ and radius a, and is surrounded by infinite fluid of density ρ_0, pressure P, and sound speed c. A pressure fluctuation $p(t)$ is then set up uniformly in space at infinity, the bubble pressure is $p_b(t)$ and the radius $R(t)$. T denotes the surface tension; $p_s(r,t)$ is the pressure induced by bubble response. Spherical symmetry is assumed, as it is known (Strasberg 1956) that negligible acoustic power is contained in any mode of oscillation of the bubble other than the symmetric expansion mode.

For the pressure drop across the bubble surface we have

$$\tilde{\omega} = (2T/a) + P \tag{3.1}$$

and

$$p_b = (2T/R) + P + p + p_s \quad \text{at} \quad r = R. \tag{3.2}$$

If adiabatic changes are assumed in the bubble,

$$p_b R^{3\gamma} = \tilde{\omega} a^{3\gamma}. \tag{3.3}$$

There is evidence to suggest that in general changes are isothermal, so that $\gamma = 1$ effectively. This is particularly likely to be true in the circumstances when the characteristic frequency of p is small compared with the bubble resonance frequency, in which case a slow forced motion of the bubble occurs. At higher frequencies, however, changes are more likely to be adiabatic, and for this reason γ is retained.

The scattered pressure p_s is a solution of the homogeneous wave equation, vanishing at $r = \infty$. Thus

$$p_s(r,t) = \frac{F(t - r/c)}{r},$$

say, so that

$$-\frac{\partial p_s}{\partial r} = \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{r}\right) p_s. \tag{3.4}$$

The gradient of p_s at $r = R$ is related to the bubble radius by the linearized equation of fluid motion,

$$-\frac{\partial p_s}{\partial r} = \rho_0 \frac{\partial q}{\partial t} = \rho_0 \frac{\partial^2 R}{\partial t^2} \quad \text{at} \quad r = R. \tag{3.5}$$

Write $R' = R - a$, and linearize (3.1)–(3.5), supposing that $|R'| \ll a$. Defining a resonance frequency ω_0 by

$$\omega_0^2 = \frac{1}{a\rho_0}\left\{(3\gamma - 1)\frac{2T}{a^2} + \frac{3\gamma P}{a}\right\}, \tag{3.6}$$

we find

$$LR' = -\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{a}\right)\left(\frac{p}{\rho_0}\right), \tag{3.7}$$

in which

$$L \equiv \frac{\partial^2}{\partial t^2} + \frac{a\omega_0^2}{c}\frac{\partial}{\partial t} + \omega_0^2.$$

We also find that

$$R' = -(1/a\rho_0\omega_0^2)\{p + p_s(a,t)\},$$

and this gives

$$Lp_s(a,t)\partial^2 = -p/\partial t^2. \tag{3.8}$$

From (3.7) we can now find an equation for the fluctuating concentration β, in the case when we have N bubbles, each of mean radius a, in unit volume of fluid. For

$$\frac{\partial \beta}{\partial t} = 4\pi a^2 N \frac{\partial R}{\partial t} = \frac{3}{a}\beta_0 \frac{\partial R}{\partial t}$$

in linearized form, where β_0 is the time-average of β. This gives

$$L\beta = -\frac{3\beta_0}{a}\left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{1}{a}\right)\left(\frac{p}{\rho_0}\right). \tag{3.9}$$

In this equation, ρ_0 may still be taken as the density of the fluid surrounding each bubble (i.e. the α-fluid) rather than of the mixture, for a small mean concentration cannot significantly alter the density when $\rho^\beta \ll \rho^\alpha$. p must apparently now include not only the forcing pressure set up at infinity, but also the resultant of all the scattered pressures set up by the distribution of bubbles. Just how much p is modified by these scattered pressures is an important point, which will receive further attention.

It will be seen later that in general the bubbles may respond significantly around two very distinct frequencies, one the resonance frequency ω_0, the other a frequency characteristic of the turbulent motion. The terms involving c in (3.9) will be found to be negligible for the forced motion at the turbulence frequency whichever of c_m, c_α is used. The problem of which value of c is relevant to (3.9) only arises in the case of resonant response, which we shall see is impossible when the bubbles are excited by nearly incompressible turbulence.

4. The sound field from forced bubble motion

We assume for the moment that the pressure field $p(t)$ generating the bubble motion is that of a turbulent flow whose internal dynamics may be regarded as nearly incompressible. Let l_0 denote a correlation scale for the turbulent flow, U the mean flow velocity and u_0 the r.m.s. turbulent velocity. The dominant frequency of the pressure field, measured in a fixed frame, is then of order U/l_0 and this is certainly an upper limit for the typical frequency of the field $p(t)$ experienced by the bubbles. Bubbles are convected with a speed of order U, and the frequencies observed following the mean flow are generally smaller than those observed at a fixed point by a factor $\sigma = u_0/U$. The dominant frequency of $p(t)$ may therefore be taken as of order u_0/l_0. Applications of flow noise theory to underwater situations commonly involve values of U of order 30 ft./sec, while $a\omega_0$ is roughly 60 ft./sec in the case of air bubbles in water at one atmosphere static pressure P. $\sigma = 5\times 10^{-2}$ is perhaps typical, and also $a \ll l_0$, for a bubble of radius comparable with l_0 could not withstand the high shear across it. It follows that $u_0/l_0 \ll \omega_0$, and we have a situation in which there is strong forcing but small response at the turbulence frequency, while at the much higher frequency ω_0 the pressure field has relatively little spectral intensity, but the bubbles have a strong intrinsic response. The response spectrum for the bubble motion therefore has two distinct peaks, near u_0/l_0 and near ω_0, corresponding to forced and resonant oscillations respectively. If the resonance peak is sufficiently narrow, we may take the two effects separately, and add them in mean-square, a conclusion which can be investigated in detail if a definite analytical form for the pressure spectrum is assumed.

For the forced motion, we neglect $\partial^2/\partial t^2$ compared with ω_0^2. The terms involving c in (3.9) can both be neglected, for they are smaller than those retained by a factor of order $u_0 a/cl_0$. This factor is extremely small even with $c = c_m$, the mixture sound speed, for c_m certainly never drops below the typical mean velocity U of order 30 ft./sec. The terms involving c represent radiation damping, and are important only in controlling the resonant response. We have, then, simply

$$\beta = -\frac{3\beta_0}{(a\omega_0)^2}\left(\frac{p}{\rho_0}\right). \tag{4.1}$$

Before using this equation in the Lighthill equation (2.3), we must first justify the assumption that the pressure field forcing any particular bubble is dominated by the eddy motion pressure. Now the mean square pressure scattered by a distribution of bubbles to any point *in* the distribution is size-dependent, and in fact varies linearly with the typical dimension L of the turbulent bubbly region. Thus, if L is large enough the scattered pressures would appear to dominate the pressure field experienced by any bubble. However, this dependence upon size L is largely irrelevant to the *problem of sound generation to distances large compared with L.* The pressure reaching a bubble from bubbles further away than a wavelength λ, approximately, is a radiating sound field pressure, and its action on the bubble is exactly that of ordinary sound waves on the bubble. The bubble is essentially passive in its response, and absorbs energy, if anything. Scattering of the incoming sound field results with a directional redistribution, and a decrease in the acoustic energy flux. The waves scattered draw their energy from the primary wave, and energies in the acoustic mode cannot be increased by the scattering. Compare Lighthill (1953), where the sound waves scattered by the interaction of a primary sound wave with turbulence draw their energy from the primary wave, and *not* from the turbulence. We can therefore reject the scattered pressures reaching a particular bubble, provided they originate at distances greater than λ from the bubble. That bubble can, however, scatter the *near-field* of any other bubble within reach (Hunter 1967), so that modifications to p from scattered pressures originating at distances less than about a wavelength λ must be considered. Whether these modifications are significant or not is now independent of the size L of the bubbly region.

This idea has important consequences for the Lighthill (1952, 1954) theory of aerodynamic noise. A turbulent eddy radiates sound waves, with a $1/r$ variation of pressure and velocity at distances greater than a wavelength. Consequently, the mean square acoustic pressure at any point in the turbulent region increases linearly with the scale L of the region, at any rate until viscous effects limit the otherwise unbounded increase which would occur in the 'compressible homogeneous turbulence' limit $L \to \infty$ (see Lighthill 1955). When L is large, but finite, one might expect these acoustic quantities to provide a significant change in the acoustic stress tensor T_{ij}, so that the sound power output from the flow might be increased. In view of the discussion above, we see that the apparent dependence of T_{ij} upon L is irrelevant to the sound generation problem. Near-field corrections to T_{ij} may be important, as an eddy can scatter the near-field of its neighbours into sound—but these corrections really should be discussed whether or not L is very large compared with l_0 or λ. The outcome of this argument appears to be that the Lighthill theory for low Mach number flows is adequate for the description of sound emission from large volumes of turbulence ($L \gg \lambda$) to *just the same extent* that it is adequate in the case $\lambda > L \gtrsim l_0$.

Returning now to the question of two-phase flow, we calculate the near-field correction to p by integrating the scattered pressure of a single bubble over the distribution of bubbles occupying a sphere of radius λ about any point in the turbulent bubbly region. The wavelength is that appropriate to propagation at frequency u_0/l_0 and at speed c_m, the low frequency sound speed in the mixture. This will be true when $L \gg \lambda$, for then the time L/c_m for propagation at speed c_m across the distance L is large compared with the time-scale l_0/u_0 of the source, and therefore the source radiates effectively into an *infinite* medium with speed c_m. On the other hand, if $L \lesssim \lambda$, the integration of the scattered pressures must run only over a sphere of radius L. The greatest modification of the pressure field then corresponds to the case $L \gg \lambda$, and then we have $\lambda \gg l_0 \gg a$, for the minimum value of c_m we shall be concerned with is 100 ft./sec, corresponding to a concentration $\beta_0 = 10^{-1}$ (see Batchelor 1967). The integration procedure is therefore relevant on two counts. In the first place, the near-field of radius λ is large enough for a continuous distribution of bubbles to be relevant, and, in the second, the near-field is so extensive that it contains many eddy volumes l_0^3. This allows us to replace each eddy by a point source of strength proportional to the eddy volume l_0^3.

The calculation is done at the end of this section, with the result

$$\langle p_s^2\rangle/\langle p^2\rangle \sim \frac{9\beta_0^2}{4\pi}\left(\frac{c_m}{u_0}\right)\left(\frac{u_0}{a\omega_0}\right)^4.$$

The brackets $\langle\ \rangle$ denote average values, all quantities being assumed stationary random functions of time. With the typical values $\beta_0 = 10^{-1}$, $c_m = 100$ ft./sec, $u_0/U = 5\times 10^{-2}$, $U = 30$ ft./sec, $a\omega_0 = 60$ ft./sec, which would seem to give the maximum value of $\langle p_s^2\rangle$ likely to occur in any practical situation, this gives

$$\langle p_s^2\rangle/\langle p^2\rangle \sim 2\times 10^{-7}.$$

Therefore it is quite adequate, for the forced motion, to assume that the pressure forcing any particular bubble is that generated by the turbulent motion alone.

We now require an estimate of the acoustic power output P_m from the region containing bubbles, whose volume is of order L^3, arising from the monopole term $\partial Q/\partial t$ in (2.3). The contribution from the forced mode only is considered here. P_m is given by

$$P_m \sim \frac{1}{4\pi\rho_0 c_\alpha}\left\langle\left(\frac{\partial Q}{\partial t}\right)^2\right\rangle l_0^3 L^3, \tag{4.2}$$

where $\rho_0 = \rho_0^\alpha$ is the density in the very distant field. This expression has been obtained from the usual retarded-potential solution

$$(\rho^\alpha - \rho_0)(\mathbf{x}, t) = \frac{1}{4\pi c_\alpha^2}\int_{\mathbf{y}} \frac{\partial Q}{\partial t}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_\alpha}\right)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \tag{4.3}$$

on the understanding that differences in retarded-time (of order l_0/c_α) corresponding to points separated by less than an eddy scale l_0 are negligible compared with the time-scale l_0/u_0 of the source field in the forced mode. This is evidently well satisfied, since the fluctuation Mach number u_0/c_α is always exceedingly small. We can express this by saying that the source field is 'acoustically compact' as far as the forced mode is concerned.

Equation (4.2) is valid only if the turbulent region has typical dimension at least of order l_0 in all directions. It is useful to write down also the power output P_Δ obtained from (4.3) when the radiating volume has the form of a sheet of area $L^2(L \gtrsim l_0)$ and thickness $\Delta \ll l_0$:

$$P_\Delta \sim \frac{1}{4\pi\rho_0 c_\alpha}\left\langle\left(\frac{\partial Q}{\partial t}\right)^2\right\rangle \Delta^2 l_0^2 L^2. \tag{4.4}$$

For the moment, however, we consider only (4.2).

Since β is small compared with unity, we have from (4.1)

$$Q = -\rho^\alpha \frac{D_\alpha}{Dt}\ln(1-\beta) \approx \rho_0 \frac{D_\alpha}{Dt}\beta = -\frac{3\beta_0}{(a\omega_0)^2}\frac{D_\alpha}{Dt}p, \tag{4.5}$$

as we are interested at present in the sound generated by fluctuations in β, rather than that generated by variations in space of the mean concentration β_0. As already discussed, p will be taken as the fluctuation in the eddy motion pressure. The magnitude of p will be estimated as the typical fluctuation in $\frac{1}{2}\rho_0\mathbf{u}^2$, i.e. $p \sim \rho_0\sigma U^2$, where $\sigma = u_0/U$ is the relative turbulence level. The time differentiation D_α/Dt will be represented by the frequency multiplication u_0/l_0. This is also the relevant estimate of the operation $\partial/\partial t$, although superficially one might expect $\partial/\partial t \sim U/l_0$. We can see this in two ways. If the derivative $\partial/\partial t$ is written as the sum of a total derivative D/Dt and a convective derivative, the total derivative is equivalent to the multiplicative operation u_0/l_0, while the convective term can be shown to represent an acoustic source of essentially lower efficiency. Alternatively, transform to a frame of reference which is convected with the mean flow. In this frame the operation $\partial/\partial t$ is certainly equivalent to multiplication by u_0/l_0, while other changes resulting from the transformation are negligible if the mean flow Mach number U/c_α is small. Either way, we see that only the *true* turbulence frequencies contribute to the acoustic power output, and that for acoustic purposes all time differentiations are equivalent to multiplication by u_0/l_0. This point is emphasized by Lighthill (1954).

With these estimates, and with neglect of convective effects, except in so far as they determine the relevant frequencies, we find that

$$P_m \sim \frac{9\beta_0^2}{4\pi}(\rho_0\sigma U^3 L^2)(\sigma M)^5\left(\frac{c_\alpha}{a\omega_0}\right)^4\left(\frac{L}{l_0}\right), \tag{4.6}$$

where $M = U/c_\alpha$. An efficiency can be conveniently defined by comparing P_m with the rate of working of the fluctuating pressure $\rho_0\sigma U^2$ against the mean flow U over an area L^2:

$$\eta_m \sim \frac{9\beta_0^2}{4\pi}(\sigma M)^5\left(\frac{c_\alpha}{a\omega_0}\right)^4\left(\frac{L}{l_0}\right). \tag{4.7}$$

The dependence of η_m on M^5 is rather surprising, being characteristic of quadrupole rather than monopole sources. It is less surprising if we remember (§ 2) that it was noted that the whole problem could be tackled using a quadrupole type of source only. The monopole Q is equivalent, in part, to $\partial(p - c_\alpha^2 \rho)/\partial t$, a quadrupole time-derivative which would occur in this alternative treatment, p and ρ now both referring to the two-phase mixture. Evidently the two forms both yield the same dependence upon M.

We have already noted that changes of volume of the bubbles are likely to take place at constant temperature when the frequency is small compared with ω_0. Thus $\gamma = 1$ effectively, and then by (3.6)

$$(a\omega_0)^2 = \{3P + (4T/a)\}/\rho_0.$$

Now, when β_0 is neither too small nor too close to unity, Batchelor's (1967) expression for the isothermal sound speed c_m at low frequencies can be written

$$c_m^2 = \{P + (4T/3a)\}/\beta_0 \rho_0, \tag{4.8}$$

and therefore we have the following simple relation between sound speed and resonance frequency,

$$(a\omega_0)^2 = 3\beta_0 c_m^2. \tag{4.9}$$

Equation (4.7) can then be written in the form

$$\eta_m \sim \frac{1}{4\pi}(\sigma M)^5 \left(\frac{c_\alpha}{c_m}\right)^4 \left(\frac{L}{l_0}\right). \tag{4.10}$$

Except for the factor $(c_\alpha/c_m)^4$, this is exactly the radiation efficiency of a typical turbulence quadrupole of strength $T_{ij} \sim \rho_0 \sigma U^2$. Note that the operation $\partial^2/\partial t^2$ on T_{ij} must be represented here by multiplication by u_0^2/l_0^2; the reasons are exactly those referred to earlier. Thus the effect of bubbles in the turbulence is to increase the acoustic power output by the factor $(c_\alpha/c_m)^4$. This increase is extremely large; in fact $(c_\alpha/c_m)^4$ is of order 10^5 even when β_0 is as small as 10^{-2}, while, for the maximum concentration $\beta_0 = 10^{-1}$ which can reasonably be encompassed by the theory, $(c_\alpha/c_m)^4$ is of order 10^7. The acoustic power output of a flow may therefore be increased by up to 70 dB by the monopole radiation of bubbles at the turbulence frequency.

To close this section, note that the pressure p_s induced by the monopole source $\partial Q/\partial t$ at any point in the turbulent bubbly region is given by

$$p_s = \frac{1}{4\pi}\int_V \left[\frac{\partial Q}{\partial t}\right]\frac{d\mathbf{y}}{r},$$

where V is the turbulent volume, and the square brackets imply evaluation at retarded-time, as in (4.3). When $u_0/c_m \ll 1$, and when the volume $V \sim L^3$ is large enough to contain many eddy volumes l_0^3, this gives

$$\langle p_s^2 \rangle \sim \frac{1}{16\pi^2}\left\langle\left(\frac{\partial Q}{\partial t}\right)^2\right\rangle l_0^3 \int_V \frac{dV}{r^2}$$

$$\sim \frac{1}{4\pi}\left\langle\left(\frac{\partial Q}{\partial t}\right)^2\right\rangle l_0^3 L. \tag{4.11}$$

Thus, as claimed earlier, the mean-square scattered pressure increases linearly with L. However, it was explained previously that, if we wish to consider the sound generation problem only, the volume integration need run only over a sphere of radius λ centred on the point considered. Hence

$$\langle p_s^2\rangle \sim \frac{1}{4\pi}\left\langle\left(\frac{\partial Q}{\partial t}\right)^2\right\rangle l_0^3\lambda,$$

and, with the estimate of $\partial Q/\partial t$ made above, we quickly find the value of $\langle p_s^2\rangle$ quoted earlier in this section.

5. Resonant response of bubbles

We have noted in the previous section that appreciable monopole radiation may result from the resonant response of bubbles to the small spectral density of the pressure field at the frequency ω_0. This, however, is a possibility which cannot occur when the applied pressure field p is that due to turbulent motion in a nearly incompressible fluid. The essential reason is that the turbulent pressure field cannot remain coherent in space, at the high frequency ω_0, over length scales as large as a bubble radius a. The spherically symmetric mode of oscillation of the bubble, which is the only mode which can give rise to volume change and so to monopole radiation, cannot then occur, for it can be created only when the pressure field has nearly the same phase at all points on the bubble surface.

The effective length scale for the turbulent field at frequency ω_0 can be found by the following argument. The bubbles travel with a translational velocity which must be comparable with the mean velocity U. Relative to the mean flow, the bubbles have fluctuating velocities which are certainly of the order of the turbulence velocity u_0 in the α-phase. The pressure fluctuations experienced by the bubbles will therefore be similar to those observed at a point following the mean flow. Now the high-frequency content of a field of turbulence, *relative* to the mean flow, occurs mainly through the convection of an almost frozen pattern of small spatial scales (i.e. small compared with l_0) by the energy-containing eddies with characteristic velocity u_0. The length scale of the pressure fluctuations at frequency ω_0 is therefore of the order of the length scale which, when convected by the large eddies at speed u_0, gives rise to the frequency ω_0. This gives $l_r \sim u_0/\omega_0$ for the 'correlation scale' at frequency ω_0 following the mean motion.

With the typical values $U = 30$ ft./sec, $a\omega_0 = 60$ ft./sec, and $u_0/U = 5\times 10^{-2}$ we then have

$$l_r/a \sim 2{\cdot}5\times 10^{-2}.$$

l_r is thus very much smaller than a, and the possibility of coherent forcing of the bubble over its entire surface is ruled out.

It might be thought that resonant response could arise if the pressure field contained an acoustic component at frequency ω_0, generated either by the turbulent eddies themselves or by their interaction with a surface in the flow. l_r would then be of the order of a wavelength λ_0 at frequency ω_0 and at the mixture sound speed c_m at frequency ω_0. The low-frequency value of c_m is 100 ft./sec when $\beta_0 = 10^{-1}$ (Batchelor 1967), and so $\lambda_0/a \sim 10$ in this case. Coherent forcing at the resonance frequency may then be possible, but the possibility is marginal, since the speed c_m at frequency ω_0 is much less than that at zero frequency. In any case, we can exclude the resonant response to small acoustic fields from the sound generation problem by the argument used in § 4. The action of sound waves on the bubble results merely in a scattering of acoustic energy, and no increase in energy output can occur. This does not quite complete the argument, for near-field scattering can occur, as we have seen. However, the scale λ_0 of the near-field in this case is very small, indeed comparable with the average separation between bubbles, so that we can probably ignore this effect, which, if it occurs at all, will depend critically on how many bubbles are in the near field at any instant.

Since we have now shown the resonant motion not to be significant, the problems of the relevant value of c in (3.9), and whether the neglect of viscous and thermal damping is valid, are of no interest here. Resonance and the dissipation which limits it are two aspects of the problem which are irrelevant when incompressible turbulence provides the excitation.

6. Radiation due to inhomogeneities in mean concentration

In the previous sections, we have considered the radiation which arises when the concentration β fluctuates about its mean value because of the compressibility of the bubbles. We now ignore that aspect of the problem and consider the radiation which may result from rapid spatial variation of the mean concentration. Situations commonly arise in which the bubbles form intense clouds, in which the concentration is high, surrounded by more or less clear fluid. It is obviously of interest to see whether the unsteady convection and distortion of these clouds can produce an appreciable sound field.

The concentration can be expressed as the sum of mean and fluctuating parts, $\beta = \bar{\beta} + \beta'$. The part of the monopole source strength involving β' has been dealt with in the last sections, and here we consider the monopole

$$\frac{\partial Q}{\partial t} \approx \rho_0 \frac{\partial}{\partial t} \frac{D_\alpha}{Dt} \bar{\beta}. \tag{6.1}$$

We shall model the cloud–water interface as a surface of discontinuity in $\bar{\beta}$ which is convected by the bubble velocity field u_i^β. The interface is taken as locally plane, so that we can write

$$\bar{\beta} = \beta_0 H(y_n - y_0), \tag{6.2}$$

where H denotes the Heaviside unit function, β_0 is the constant value of the mean concentration within the cloud, y_n the co-ordinate normal to the interface, $y_0(t)$ the y_n co-ordinate of the interface at time t. We have

$$\frac{D_\beta}{Dt} \bar{\beta} = \left(\frac{\partial}{\partial t} + u_j^\beta \frac{\partial}{\partial y_j} \right) \bar{\beta} = 0 \tag{6.3}$$

and

$$\frac{dy_0(t)}{dt} = u_n^\beta.$$

This gives

$$\frac{D_\alpha}{Dt} \bar{\beta} = (u_i^\alpha - u_i^\beta) \frac{\partial \bar{\beta}}{\partial y_i},$$

and then the monopole in (6.1) can be conveniently combined with that part of the momentum defect dipole $G_i' = \rho_0 \partial\beta u_i^\alpha/\partial t$ which contains $\bar{\beta}$, to yield

$$\frac{\partial Q}{\partial t}-\frac{\partial G_i'}{\partial y_i} = -\rho_0\frac{\partial}{\partial y_i}\frac{\partial}{\partial t}(u_i^\beta\bar{\beta})-\rho_0\frac{\partial}{\partial t}\left\{\bar{\beta}\frac{\partial}{\partial y_i}(u_i^\alpha-u_i^\beta)\right\}. \tag{6.4}$$

The first term in (6.4) represents a dipole field arising from the random distortion of the interface by the motion of the bubbles. This dipole term will be considered further below. For the monopole term in (6.4) we estimate the divergences of u_i^α, u_i^β from (2.1) and from the analogous equation

$$\frac{\partial}{\partial t}\beta\rho^\beta+\frac{\partial}{\partial y_i}\beta\rho^\beta u_i^\beta = 0, \tag{6.5}$$

expressing conservation of the mass of the β-phase. Neglecting small variations in ρ^α we have

$$\frac{\partial u_i^\alpha}{\partial y_i}\approx\frac{D_\alpha}{Dt}\beta,\quad \frac{\partial u_i^\beta}{\partial y_i}\approx-\frac{1}{\beta}\frac{D_\beta}{Dt}\beta,$$

and the latter term dominates, since $\beta \ll 1$. Then

$$\begin{aligned}\bar{\beta}\frac{\partial}{\partial y_i}(u_i^\alpha-u_i^\beta) &= \bar{\beta}\frac{1}{\beta_0}\frac{D_\beta}{Dt}\beta' \\ &= -\frac{3\bar{\beta}}{(a\omega_0)^2}\frac{D_\beta}{Dt}p,\end{aligned}$$

where (4.1) has been used to relate the fluctuating concentration β' to the pressure p. The monopole term in (6.4) then becomes

$$\left.\begin{aligned}&-\frac{3\beta_0}{(a\omega_0)^2}\left(\frac{D_\beta}{Dt}p\right)u_n^\beta\,\delta(y_n-y_0)\\ &+\frac{3\beta_0}{(a\omega_0)^2}\left(\frac{\partial}{\partial t}\frac{D_\beta}{Dt}p\right)H(y_n-y_0).\end{aligned}\right\} \tag{6.6}$$

The operations D_β/Dt and D_α/Dt on the pressure p are equivalent, to the degree of accuracy possible here. Comparing the second term of (6.6) with the value of $\partial Q/\partial t$ obtained from (4.5), we see that this term involving the H-function represents the monopole sources distributed throughout the interior of the cloud. On the other hand, the sources represented by the first term of (6.6) are confined to the interface between the cloud and the clear fluid outside it. The interface is equivalent to a distribution of surface sources, of strength

$$\left.\begin{aligned}&\frac{3\beta_0}{(a\omega_0)^2}\left(\frac{D_\beta}{Dt}p\right)u_n^\beta\\ &\sim\frac{3\beta_0\rho_0u_0^3U}{(a\omega_0)^2l_0}\ \text{per unit area.}\end{aligned}\right\} \tag{6.7}$$

The resulting efficiency of these sources is found to be equal to that produced by the monopoles distributed in a sheet whose total area is that of the interface and whose thickness is just one eddy length l_0.

Thus an area S of the interface produces the same power output as do the monopole sources distributed throughout a volume $l_0 S$. The surface effects are therefore extremely large when the typical dimension of the bubbly region is comparable with l_0. If the region is in the form of a thin sheet of thickness $\Delta \lesssim l_0$, the surface sources will dominate the radiation field. In that case, (4.2) represents an overestimate of the volume-induced sound, and (4.4) should then be used.

The dipole term in (6.4) can also be expressed in terms of surface and volume distributions. Taking the dominant surface source term, the radiated density field can be shown to be given by

$$(\rho^\alpha - \rho_0)(\mathbf{x}, t) = \frac{\rho_0 \beta_0}{4\pi c_\alpha^3}\left(\frac{x_i}{r^2}\right)\int_S \left[u_i^\beta u_n^\beta \frac{\partial u_n^\beta}{\partial y_n}\right] dS, \tag{6.8}$$

from which the radiation efficiency follows as

$$\eta_S \sim (\beta_0^2/4\pi)\,\sigma M^3.$$

The ratio of this efficiency to that of the surface monopoles is of order

$$\tfrac{1}{9}\sigma^{-4} M^{-2}\left(\frac{a\omega_0}{c_\alpha}\right)^4,$$

and this is slightly greater than unity when the typical values given in § 4 are again used. Therefore this form of radiation is also important when the radiating volume is in the form of a sheet with thickness less than about l_0. The dipole and monopole sound fields are comparable in this case essentially because the dipole field exists independently of the response of the bubbles, and would be produced even if the bubbles were rigid and could not respond. On the other hand, the monopole surface sound field depends almost entirely on the compliance of the bubbles, and the velocities induced by bubble response are small compared with those in the turbulence which provide the convection and distortion of the interface, and hence the dipole surface sound.

7. Dipole sources of sound

The term $G_i' = \partial \beta \rho^\alpha u_i^\alpha / \partial t$ contains contributions other than those arising from changes in mean concentration. We have, identically,

$$G_i' = \frac{\rho^\alpha u_i^\alpha}{1-\beta}\frac{\partial \beta}{\partial t} + \frac{\beta}{1-\beta}\frac{\partial}{\partial t}(1-\beta)\rho^\alpha u_i^\alpha.$$

Assume that β is small compared with unity, and use the momentum equation (2.2) to transform the last term above. Apart from the interphase force, we have

$$G_i' = \rho^\alpha u_i^\alpha \frac{D_\alpha}{Dt}\beta + p_{ij}\frac{\partial \beta}{\partial x_j} - \frac{\partial}{\partial x_j}\{\beta \rho^\alpha u_i^\alpha u_j^\alpha + \beta p_{ij}\}. \tag{7.1}$$

The last term in (7.1) represents a quadrupole source, whose strength certainly vanishes in the far-field where $\beta = 0$. It therefore represents a basically less efficient source than do the other terms, and may be neglected. From the estimates $D_\alpha/Dt \sim u_0/l_0$, $\partial/\partial x_j \sim 1/l_0$, $p_{ij} \sim \rho_0 \sigma U^2$ we see that the remaining terms are of the same order of magnitude. We use these estimates, with (4.1) to relate β to p, to obtain the dipole efficiency η_d due to volumetric response of the bubbles:

$$\eta_d \sim \frac{9\beta_0^2}{4\pi}\sigma^5 M^7 \left(\frac{c_\alpha}{a\omega_0}\right)^4 \left(\frac{L}{l_0}\right) = M^2\eta_m, \tag{7.2}$$

where η_m is the monopole efficiency given in (4.10). The factor M^2 ensures that this kind of radiation is negligible in all cases.

Neglect of the interphase force compared with the displaced momentum is certainly valid for the case of air bubbles in water, since the bubbles have appreciable volume but negligible mass. If, however, the density of the β-phase is large compared with that of the α-phase, the interphase force may be important. This happens in the case when the α-phase is a gas, and the β-phase a distribution of rigid dust particles. The volume concentration of dust particles is supposed negligible, though the mass concentration may be appreciable. We obtain the case of a dusty gas from our general equations by letting $\beta \to 0$, $\rho^\beta \to \infty$, so that the mass concentration $\beta\rho^\beta/\rho^\alpha$ has a finite limit, f say. The terms $\partial Q/\partial t$ and $\partial G_i'/\partial x_i$ now vanish identically, and the influence of the dust particles on the gas is contained entirely in the interphase force F_i.

Suppose that the dust particle number density is N, and that each particle has mass m, so that $f\rho^\alpha = Nm$. Saffman (1962) wrote down the equations of dusty gas flow, and assumed that the force density F_i was given by a linear Stokes law,

$$F_i = KN(u_i^\beta - u_i^\alpha). \tag{7.3}$$

u_i^β is the velocity of the β-phase at $(\mathbf{x}, t)$ and K is a constant proportional to the viscosity of the α-phase and to the typical particle dimension. This viscous drag force is very much larger than any forces due to virtual inertia for the kinds of system envisaged by Saffman. We do not need the specific form (7.3) here, though it is useful in that it allows us to define a relaxation time for the dust particles as $\tau = m/K$. In most practical cases τ is small compared with the characteristic time of the gas motion, and, when this is so, the dust particles follow the gas motion closely. The effect of the dust particles is then to increase the effective density of the mixture from ρ^α to $(1+f)\rho^\alpha$ without change in the other variables. In particular, the sound speed c_m in the dusty gas is given by

$$c_m^2 = c_\alpha^2/(1+f_0), \tag{7.4}$$

where f_0 denotes an average value of f. This result is true, irrespective of the validity of (7.3), provided only that a suitable relaxation time is small compared with the time-scale of the gas motion.

Now the momentum and mass conservation equations for the dust particles may be written (Saffman 1962)

$$\frac{\partial}{\partial t} f\rho^\alpha u_i^\beta + \frac{\partial}{\partial x_j} f\rho^\alpha u_i^\beta u_j^\beta = -F_i, \tag{7.5}$$

$$\frac{\partial}{\partial t} f\rho^\alpha + \frac{\partial}{\partial x_i} f\rho^\alpha u_i^\beta = 0. \tag{7.6}$$

Using these equations, the dipole and quadrupole sources in (2.3) may be expressed as

$$-\frac{\partial F_i}{\partial x_i}+\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = -\frac{\partial^2}{\partial t^2} f\rho^\alpha + \frac{\partial^2 W_{ij}}{\partial x_i \partial x_j}, \tag{7.7}$$

in which

$$W_{ij} = \rho^\alpha u_i^\alpha u_j^\alpha + f\rho^\alpha u_i^\beta u_j^\beta + p_{ij} - c_\alpha^2 \rho^\alpha \delta_{ij}.$$

To regard the terms in (7.7) as a monopole and a quadrupole, respectively, would be an error. For the monopole term may be rewritten as

$$-\frac{\partial^2}{\partial t^2}(1+f)\rho^\alpha + \frac{\partial^2}{\partial t^2}\rho^\alpha$$

$$= -\frac{1}{c_m^2}\frac{\partial^2 p}{\partial t^2} + \frac{1}{c_\alpha^2}\frac{\partial^2 p}{\partial t^2} = -\frac{f_0}{c_\alpha^2}\frac{\partial^2 p}{\partial t^2},$$

since, by definition,

$$\frac{1}{c_m^2} = \frac{\partial}{\partial p}(1+f)\rho^\alpha.$$

(The low relaxation time limit has been assumed.) Therefore the monopole strength is $O(M^2)$, rather than $O(1)$, and this monopole is equivalent to an isotropic $O(1)$ quadrupole.

We can now estimate the efficiency corresponding to the two terms in (7.7), remembering that in the low relaxation time limit we have $u_i^\alpha \approx u_i^\beta$, and that ρ^α is increased to $(1+f)\rho^\alpha$. This applies also to the factor $\rho_0 \sigma U^3 L^2$ used to normalize the efficiency. We find that

$$\eta_f \sim \frac{1}{4\pi}(\sigma M)^5 f_0^2 (1+f_0)\left(\frac{L}{l_0}\right), \tag{7.8}$$

$$\eta_w \sim \frac{1}{4\pi}(\sigma M)^5 (1+f_0)\left(\frac{L}{l_0}\right), \tag{7.9}$$

for the efficiencies corresponding to the first and second terms on the right of (7.7), respectively. When $f_0 < 1$, $\eta_w > \eta_f$, and then, in virtue of (7.4),

$$\eta \sim \frac{1}{4\pi}(\sigma M)^5 \left(\frac{c_\alpha}{c_m}\right)^2 \left(\frac{L}{l_0}\right). \tag{7.10}$$

The radiation efficiency is increased by the factor $(c_\alpha/c_m)^2$ by the presence of the dust, and the radiated power is increased by the factor $(c_\alpha/c_m)^4$, exactly as in the case of a suspension of air bubbles in water. However the increases are negligible, in practical terms, when $f_0 < 1$. When $f_0 > 1$, we have

$$\eta \sim \frac{1}{4\pi}(\sigma M)^5 \left(\frac{c_\alpha}{c_m}\right)^6 \left(\frac{L}{l_0}\right), \tag{7.11}$$

so that now the efficiency is increased by the sixth power of the sound speed ratio, and the power output by the eighth power. If the typical velocity U is the same for both a clean and a dusty gas, this increase in power output is large, up to about 20 dB perhaps, for mass concentrations f_0 of the order of 2 or 3 which are common in many industrial processes where dusty gases are used to increase rates of heat transfer. In some cases, however, this comparison is not relevant.

For example, if the mechanical power of the flow were the same for the clean and dusty gases, as might be the case in a jet-type flow, then

$$(1+f_0)\,U^3 = U_0^3,$$

where U, U_0 are the values of the same typical velocity with and without the presence of dust, respectively. The increase in power output, according to (7.11), would then only be of order $f_0^{\frac{7}{3}}$ instead of f_0^4, but should still provide an effect which is easily detectable in practice.

8. Conclusions

The radiation properties of turbulent flow in water have been shown to be greatly modified by the presence of a small distribution of air bubbles in the turbulence. In the model used here to describe this process, the effects of the bubbles have been represented as acoustically equivalent to a volume distribution of monopoles and dipoles, in addition to the quadrupoles equivalent to the fluctuating stresses in the turbulence. Monopole radiation results from the low-frequency forced volumetric response of the bubbles to the turbulent pressure field. The effect of this radiation is in all cases equivalent to an increase in the quadrupole radiation (above its value in pure water) by the factor $(c_\alpha/c_m)^4$, where c_α, c_m are the sound speeds in pure water and in the bubbly region respectively. The acoustic power output of the flow is increased by 50 dB for a 1 % air/water concentration, and by 70 dB for a 10 % concentration. These may be regarded as relevant figures for many practical situations.

Significant volumetric response of the bubbles at their high natural resonance frequency has been shown to be impossible when the excitation is due to nearly incompressible turbulence alone. The reason is that the length scale over which the pressure field remains coherent at the resonance frequency is found to be very small compared with the bubble radius. The exclusion of resonant response indicates that the use of linear equations to represent the bubble response is justified.

Dipole radiation arises through the displacement of fluid momentum by the gas bubbles, and through the action of the force between bubbles and fluid. The momentum displacement effect is the dominant cause of dipole radiation, but the resulting efficiency is always negligible compared with that of the monopoles.

Monopole and dipole radiation occur through the unsteady convection of the interface between the bubble/water mixture and the clear fluid outside it. In this case the radiation is generated essentially by a distribution of sources over the interface. The monopole and dipole radiation efficiencies are comparable, and are important, compared with the volume-generated sound, if the radiating volume is in the form of a sheet with thickness equal to, or smaller than, an eddy scale l_0. If the thickness is equal to the eddy scale, which may be of the order of one foot in practical situations, then surface and volume monopole power outputs are equal, and either overwhelms the radiation which would occur if no bubbles are present.

Finally, in the case of a suspension of dust particles in a gas, no monopole sound can be produced. Dipole radiation occurs through the action of the force exerted by the dust on the gas, and it is shown that this form of radiation is equivalent to an amplification of the quadrupole sound which occurs in a clean gas. When the mass concentration of dust exceeds unity, this increase is large—up to about 20 dB perhaps, though not nearly as large as that provided by the presence of bubbles. Moreover, the presence of a large mass concentration of dust will substantially reduce the flow speeds if the flow is governed by a source of constant power. In that case, the quadrupole sound is enhanced, in intensity, by the factor (mass concentration $f_0)^{\frac{7}{3}}$ over its value in a clean gas under the same mechanical power. This would still indicate that the use of dust particles in many industrial processes will make a considerable contribution to the noise level.

This research was carried out under the Naval Ship Systems Command General Hydromechanics Research Program, administered by the Naval Ship Research and Development Center under Contract N 62558-4996.

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